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EXISTENCE OF GLOBAL SOLUTIONS FOR A SYSTEM OF REACTION-DIFFUSION EQUATIONS HAVING A FULL MATRIX

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Abstract. In this paper we generalize a result obtained in [8] concerning uniform boundedness and so the global existence of solutions for reaction-diffusion systems with a general full matrix of diffusion coefficients. Our techniques are based on invariant regions and Lyapunov functional methods.

1. introduction

We are interested in global existence in time of solutions to the reaction- diffusion systems of the form

$$(1.1) \quad \frac{\partial u}{\partial t} - a\Delta u - b\Delta v = f(u, v) \quad \text{in }]0, +\infty[\times \Omega$$

$$(1.2) \quad \frac{\partial v}{\partial t} - c\Delta u - d\Delta v = g(u, v) \quad \text{in }]0, +\infty[\times \Omega$$

with the following boundary conditions

$$(1.3) \quad \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = 0 \quad \text{in }]0, +\infty[\times \partial\Omega$$

and the initial data

$$(1.4) \quad u(0, x) = u_0, \quad v(0, x) = v_0 \quad \text{in } \Omega,$$

where Ω is an open bounded domain of class C^1 in \mathbb{R}^n with boundary $\partial\Omega$, $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$, Δ denotes the Laplacian operator

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with respect to the x variable, a, b, c, d are positive constants satisfying the condition $(b + c)^2 < 4ad$ which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion is positive definite. The eigenvalues λ_1 and λ_2 ($\lambda_1 < \lambda_2$) of the matrix of diffusion are positive. If we assume that $a < d$, then we have

$$\lambda_1 < a < d < \lambda_2.$$

The initial data are assumed to be in the following region

$$\Sigma = \left\{ (u_0, v_0) \in \mathbb{R}^2 \text{ such that } \frac{a - \lambda_2}{c} v_0 \leq u_0 \leq \frac{a - \lambda_1}{c} v_0 \right\}.$$

We suppose that the reaction terms f and g are continuously differentiable on Σ , $(f(r, s), g(r, s))$ is in Σ , for all (r, s) in $\partial\Sigma$ (we say that (f, g) points into Σ on $\partial\Sigma$), i.e.,

$$(1.5) \quad \frac{a - \lambda_2}{c} g\left(\frac{a - \lambda_2}{c} s, s\right) \leq f\left(\frac{a - \lambda_2}{c} s, s\right) \text{ and } f\left(\frac{a - \lambda_1}{c} s, s\right) \leq \frac{a - \lambda_1}{c} g\left(\frac{a - \lambda_1}{c} s, s\right),$$

for all $s \geq 0$.

See [5, 12], for more details.

Assume further that

$$(1.6) \quad \sup\left(\left| -f + \frac{a - \lambda_1}{c} g \right|(r, s), \left| f - \frac{a - \lambda_2}{c} g \right|(r, s)\right) \leq C(|r| + s + 1)^m, \forall r, s \in \Sigma,$$

where C is a positive constant and $m \geq 1$.

We suppose that one of the following conditions is satisfied:

1-There exist $p \geq 2$; $c(p) > 0$ and positive numbers $(B_i(p))_{0 \leq i \leq p}$ such that

$$(1.7) \quad (B_{i-1}(p) - B_i(p)) f(r, s) + \left(\frac{a - \lambda_1}{c} B_i(p) - \frac{a - \lambda_2}{c} B_{i-1}(p)\right) g(r, s) \leq C(p)(r + s + 1),$$

where

$$(1.8) \quad (a + d)^2 B_i^2(p) \leq 4.(a.d - bc) B_{i-1}(p) B_{i+1}(p).$$

2-There exist $c(1) > 0$ and $B_i(1)$, $0 \leq i \leq 1$ such that

$$(1.9) \quad (B_0(1) - B_1(1)) f(r, s) + \left(\frac{a - \lambda_1}{c} B_1(1) - \frac{a - \lambda_2}{c} B_0(1)\right) g(r, s) \leq C(1)(r + s + 1).$$

N. Alikakos [1] established global existence and L^∞ -bounds of solutions for positive initial data for $b = c = 0$, $f(u, v) = -g(u, v) = -uv^\beta$ and $1 < \beta < (n + 2)/n$. In

[6] K. Masuda showed that solutions to this system exist globally for every $\beta > 1$. A. Haraux and A. Youkana [3] generalized the method of K. Masuda with the reactions term

$$f(u, v) = -g(u, v) = -u\varphi(v),$$

where φ is a nonnegative function satisfying the following condition

$$\lim_{v \rightarrow +\infty} \frac{\log(1 + \varphi(v))}{v} = 0.$$

The components $u(t, x)$ and $v(t, x)$ represent either chemical concentrations or biological population densities and the system (1.1)–(1.4) is a mathematical model describing various chemical and biological phenomena (see E. L. Cussler [2], J. Savchik [11]).

The present investigation is a continuation of results obtained in [8, 9]. In this study, we will treat the case of a general full matrix of diffusion coefficients.

2. Existence of local solutions

The usual norms in spaces $L^p(\Omega)$, $L^\infty(\Omega)$ and $C(\overline{\Omega})$ are respectively denoted by

$$\|u\|_p^p = \frac{1}{|\Omega|} \int_{\Omega} |u(x)|^p dx$$

$$\|u\|_\infty = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|$$

$$\|u\|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |u(x)|$$

For any initial data in $C(\overline{\Omega})$ or $L^p(\Omega)$, $p \in [1, \infty[$ local existence and uniqueness of solutions to the initial value problem (1.1)–(1.4) follow from the basic existence theory for abstract semilinear differential equations (see D. Henry [4] and A. Pazy [10]). The solutions are classical on $]0, T^*[$, where T^* denotes the eventual blowing-up time in $L^\infty(\Omega)$.

Furthermore, if $T^* < +\infty$, then

$$\lim_{t \uparrow T^*} (\|u(t)\|_\infty + \|v(t)\|_\infty) = +\infty.$$

Therefore, if there exists a positive constant C such that

$$\|u(t)\|_\infty + \|v(t)\|_\infty \leq C, \forall t \in]0, T^*[,$$

then $T^* = +\infty$.

3. Existence of global solutions

Multiplying equation (1.2) one time through by $\frac{a-\lambda_1}{c}$ and subtracting (1.1) and another time by $-\frac{a-\lambda_2}{c}$ and adding (1.1), we get

$$(3.1) \quad \frac{\partial w}{\partial t} - \lambda_1 \Delta w = F(w, z) \quad \text{in }]0, T^*[\times \Omega,$$

$$(3.2) \quad \frac{\partial z}{\partial t} - \lambda_2 \Delta z = G(w, z) \quad \text{in }]0, T^*[\times \Omega,$$

with the boundary conditions

$$(3.3) \quad \frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0 \quad \text{in }]0, T^*[\times \partial\Omega,$$

and the initial data

$$(3.4) \quad w(0, x) = w_0(x), z(0, x) = z_0(x) \quad \text{in } \Omega,$$

where

$$(3.5) \quad \begin{aligned} w(t, x) &= -u(t, x) + \frac{a-\lambda_1}{c} v(t, x), \\ z(t, x) &= u(t, x) - \frac{a-\lambda_2}{c} v(t, x), \end{aligned}$$

for any $(t, x) \text{ in }]0, T^*[\times \Omega$ and

$$(3.6) \quad \begin{aligned} F(w, z) &= (-f + \frac{a-\lambda_1}{c} g)(u, v), \\ G(w, z) &= (f - \frac{a-\lambda_2}{c} g)(u, v), \text{ for all } (u, v) \in \Sigma. \end{aligned}$$

To prove that the solutions of (1.1)-(1.4) are global, comes back in even to prove it for problem (3.1)-(3.4). To this subject, it is well known that, it is sufficient to derive a uniform estimate of the quantity

$$\sup(\|(F(w, z))\|_q, \|(G(w, z))\|_q),$$

on $]0, T^*[$ for some $q > \frac{n}{2}$.

Now, we present the main result

Theorem 3.1. *Let $(w(t, \cdot), z(t, \cdot))$ be a solution of (3.1)-(3.4). If one of the conditions (1.7) or (1.9) has been satisfied, there would exist an integer $p \geq 1$ and a continuous function $c_p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that*

$$\sup(\|(w(t, \cdot), z(t, \cdot))\|_p, \|z(t, \cdot)\|_p) \leq c_p(t), t < T^*.$$

Proof. We consider the functional

$$(3.7) \quad L_p(t) = \int_{\Omega} \left(\sum_{i=0}^{i=p} C_p^i B_i(p) w^i z^{p-i} \right) dx = \int_{\Omega} \left(\sum_{i=0}^{i=p} \alpha_i(p) w^i z^{p-i} \right) dx,$$

where

$$\alpha_i(p) = C_p^i B_i(p), i = 0, \dots, p.$$

Differentiating L_p with respect to t , we obtain

$$L'_p(t) = \int_{\Omega} \left(\sum_{i=1}^{i=p} i \alpha_i(p) w^{i-1} z^{p-i} \right) \frac{\partial w}{\partial t} dx + \int_{\Omega} \left(\sum_{i=0}^{i=p-1} (p-i) \alpha_i(p) w^i z^{p-i-1} \right) \frac{\partial z}{\partial t} dx.$$

Consequently,

$$L'_p(t) = \int_{\Omega} \left(\sum_{i=1}^{i=p} i \alpha_i(p) w^{i-1} z^{p-i} \right) \frac{\partial w}{\partial t} dx + \int_{\Omega} \left(\sum_{i=1}^{i=p} (p-i+1) \alpha_{i-1}(p) w^{i-1} z^{p-i} \right) \frac{\partial z}{\partial t} dx.$$

using (3.1) and (3.2), we get

$$\begin{aligned} L'_p(t) = & \int_{\Omega} \left(\sum_{i=1}^{i=p} i \alpha_i(p) w^{i-1} z^{p-i} \right) (F(w, z) + \lambda_1 \Delta w) dx + \\ & + \int_{\Omega} \left(\sum_{i=1}^{i=p} (p-i+1) \alpha_{i-1}(p) w^{i-1} z^{p-i} \right) (G(w, z) + \lambda_2 \Delta z) dx. \end{aligned}$$

which implies

$$\begin{aligned}
(3.8) \quad L'_p(t) &= \int_{\Omega} \sum_{i=1}^{i=p} i\alpha_i(p) F(w, z) w^{i-1} z^{p-i} dx + \\
&\int_{\Omega} \left(\sum_{i=1}^{i=p} (p-i+1) \alpha_{i-1}(p) G(w, z) w^{i-1} z^{p-i} dx + \right. \\
&+ \int_{\Omega} \sum_{i=1}^{i=p} (\lambda_1 i \alpha_i(p) \Delta w) w^{i-1} z^{p-i} dx + \\
&\left. + \int_{\Omega} \sum_{i=1}^{i=p} (\lambda_2 (p-i+1) \alpha_{i-1}(p) \Delta z) w^{i-1} z^{p-i} dx. \right.
\end{aligned}$$

We distinguish two cases:

1-when $p = 1$, we obtain from (3.8)

$$\begin{aligned}
L'_1(t) &= \int_{\Omega} (\lambda_1 \alpha_1(1) \Delta w + \lambda_2 \alpha_0(1) \Delta z) dx + \\
&\int_{\Omega} (\alpha_1(1) F(w, z) + \alpha_0(1) G(w, z)) dx,
\end{aligned}$$

By applying Green's formula, we have

$$L'_1(t) = \int_{\Omega} (B_1(1) F(w, z) + B_0(1)) G(w, z) dx,$$

Using conditions (1.9), (3.5) and (3.6), we deduce

$$\begin{aligned}
L'_1(t) &\leq c'_0 \int_{\Omega} (w + z + 1) dx = \\
&c'_0 \int_{\Omega} (w + z) dx + c'_0 \text{mes}(\Omega),
\end{aligned}$$

It follows that

$$(3.9) \quad L'_1(t) \leq c_1 L_1(t) + c_2, \quad t < T^*,$$

where

$$\begin{aligned}
c_1 &= c'_0 \max(B_1(1), B_0(1)), \\
c_2 &= c'_0 \text{mes}(\Omega).
\end{aligned}$$

By a simple integration of (3.9) for all $t < T^*$, we have

$$L_1(t) \leq (L_1(0) + \frac{c_2}{c_1}) \exp(c_1 t) - \frac{c_2}{c_1},$$

from (3.7), we obtain

$$L_1(t) \geq \min(\alpha_1(1), \alpha_0(1)) \int_{\Omega} (w + z) dx \geq \min(\alpha_1(1), \alpha_0(1)) \sup \|w(t, \cdot), z(t, \cdot)\|_1,$$

Then, we get

$$\sup \|w(t, \cdot), z(t, \cdot)\|_1 \leq c_1(t), \text{ for } t \leq T^*,$$

where

$$c_1(t) = \frac{1}{\min(\alpha_1(1), \alpha_0(1))} \times \left\{ (L_1(0) + \frac{c_2}{c_1}) \exp(c_1 t) - \frac{c_2}{c_1} \right\},$$

2-If $p \geq 2$

We set

$$(3.10) \quad T = \int_{\Omega} \sum_{i=1}^{i=p} (\lambda_1 i \alpha_i(p) \Delta w) w^{i-1} z^{p-i} dx + \int_{\Omega} \sum_{i=1}^{i=p} (\lambda_2 (p-i+1) \alpha_{i-1}(p) \Delta z) w^{i-1} z^{p-i} dx.$$

The inequality (3.10) can be written as

$$T = \sum_{i=1}^{i=p} \int_{\Omega} \Delta (\lambda_1 i \alpha_i(p) w) w^{i-1} z^{p-i} dx + \sum_{i=1}^{i=p} \int_{\Omega} \Delta (\lambda_2 (p-i+1) \alpha_{i-1}(p) z) w^{i-1} z^{p-i} dx.$$

By a simple use of Green's formula, we obtain

$$T = - \sum_{i=1}^{i=p} \int_{\Omega} (\nabla (\lambda_1 i \alpha_i(p) w) \nabla (w^{i-1} z^{p-i})) dx - \sum_{i=1}^{i=p} \int_{\Omega} \nabla (\lambda_2 (p-i+1) \alpha_{i-1}(p) z) \nabla (w^{i-1} z^{p-i}) dx.$$

Which implies

$$\begin{aligned}
T = & - \int_{\Omega} \sum_{i=2}^{i=p} \lambda_1 i(i-1) \alpha_i(p) w^{i-2} z^{p-i} \nabla^2 w dx + \\
& \int_{\Omega} \sum_{i=1}^{i=p-1} \lambda_1 i(p-i) w^{i-1} z^{p-i-1} \nabla w \nabla z dx + \\
& \int_{\Omega} \sum_{i=2}^{i=p} \lambda_2 (i-1)(p-i+1) \alpha_{i-1}(p) w^{i-2} z^{p-i} \nabla w \nabla z dx + \\
& \int_{\Omega} \sum_{i=1}^{i=p-1} \lambda_2 (p-i+1)(p-i) \alpha_{i-1}(p) w^{i-1} z^{p-i-1} \nabla^2 z dx,
\end{aligned}$$

By a simple computation and from (3.8), it follows that

$$\begin{aligned}
L'_p(t) = & \int_{\Omega} \sum_{i=1}^{i=p} (i C_p^i B_i(p)) F(w, z) w^{i-1} z^{p-i} dx + \\
& \int_{\Omega} \left(\sum_{i=1}^{i=p} (p-i+1) C_p^{i-1} B_{i-1}(p) G(w, z) w^{i-1} z^{p-i} \right) dx - \\
& \int_{\Omega} \left(\sum_{i=1}^{i=p-1} \left\{ \begin{array}{c} \lambda_1 i(i+1) C_p^{i+1} B_{i+1}(p) \nabla^2 w + (a+d) i(p-i) C_p^i B_i(p) \nabla w \nabla z + \\ \lambda_2 (p-i)(p-i+1) C_p^{i-1} B_{i-1}(p) \nabla^2 z \end{array} \right\} w^{i-1} z^{p-i-1} \right) dx,
\end{aligned}$$

Using the fact that

$$\begin{aligned}
i C_p^i &= (p-i+1) C_p^{i-1} = p C_{p-1}^{i-1}, \\
i(i+1) C_p^{i+1} &= i(p-i) C_p^i = (p-i)(p-i+1) C_p^{i-1} = p(p-1) C_{p-2}^{i-1},
\end{aligned}$$

we conclude

$$\begin{aligned}
L'_p(t) = & \int_{\Omega} \left(\sum_{i=1}^{i=p} (p C_{p-1}^{i-1} [B_i(p) F(w, z) + B_{i-1}(p) G(w, z)] w^{i-1} z^{p-i}) \right) dx - \\
& p(p-1) \int_{\Omega} \left(\sum_{i=1}^{i=p-1} C_{p-2}^{i-1} \times [\lambda_1 B_{i+1}(p) \nabla^2 w + (a+d) B_i(p) \nabla w \nabla z + \lambda_2 B_{i-1}(p) \nabla^2 z] w^{i-1} z^{p-i-1} \right) dx.
\end{aligned}$$

From (1.8), it follows that the quadratic forms

$$\lambda_1 B_{i+1}(p) \nabla^2 w + (a+d) B_i(p) \nabla w \nabla z + \lambda_2 B_{i-1}(p) \nabla^2 z,$$

are positive since

$$((a + d)B_i(p))^2 - 4\lambda_1 B_{i+1}\lambda_2 B_{i-1} = (a + d)^2 (B_i(p))^2 - 4(ad - bc)B_{i+1}B_{i-1} \leq 0,$$

Consequently,

$$L'_p(t) \leq p \int_{\Omega} \sum_{i=1}^{i=p} C_{p-1}^{i-1} [B_i(p)F(w, z) + B_{i-1}(p)G(w, z)] w^{i-1} z^{p-i} dx.$$

Using conditions (1.7), (3.5) and (3.6), we get for an appropriate constant $c_0(p)$

$$L'_p(t) \leq c_0(p) \int_{\Omega} \left(\sum_{i=1}^{i=p} C_{p-1}^{i-1} (w + z + 1) w^{i-1} z^{p-i} \right) dx,$$

By a simple computation, we conclude

$$(3.11) \quad L'_p(t) \leq c_0(p) \left(\int_{\Omega} \sum_{i=0}^{i=p} C_p^i w^i z^{p-i} dx + \int_{\Omega} \left(\sum_{i=0}^{i=p-1} C_{p-1}^i w^i z^{p-i-1} \right) dx \right),$$

Using the fact that

$$\sum_{i=0}^{i=p-1} C_{p-1}^i w^i z^{p-i-1} = (w + z)^{p-1},$$

the inequality (3.11) can be written as

$$L'_p(t) \leq c_1(p)L_p(t) + c_0(p) \int_{\Omega} (w + z)^{p-1} dx,$$

Applying Hölder's inequality to the second term in the right hand side of the above inequality, we obtain

$$L'_p(t) \leq c_1(p)L_p(t) + c_0(p)(\text{mes}(\Omega))^{\frac{1}{p}} \left(\int_{\Omega} (w + z)^p dx \right)^{\frac{p-1}{p}},$$

Since the following inequality holds,

$$(w + z)^p = \sum_{i=0}^{i=p} C_p^i w^i z^{p-i} \leq \frac{\sup_{0 \leq i \leq p} C_p^i}{\min_{0 \leq i \leq p} \alpha_i(p)} \sum_{i=0}^{i=p} \alpha_i(p) w^i z^{p-i}$$

We conclude that the functional L_p satisfies the following differential inequality

$$L'_p(t) \leq c_1(p)L_p(t) + c_2(p)(L_p(t))^{\frac{p-1}{p}}, \forall t < T^*,$$

where

$$c_2(p) = c_0(p)(\text{mes}(\Omega))^{\frac{1}{p}} \left(\frac{\sup_{1 \leq i \leq p} C_p^i}{\min_{1 \leq i \leq p} \alpha_i(p)} \right)^{\frac{p-1}{p}},$$

while putting

$$q(t) = (L_p(t))^{\frac{1}{p}}.$$

One gets

$$pq'(t) \leq c_1(p)q(t) + c_2(p),$$

which gives us, by a simple integration

$$(3.12) \quad (L_p(t))^{\frac{1}{p}} \leq \left[(L_p(0))^{\frac{1}{p}} + \frac{c_2(p)}{c_1(p)} \right] \exp\left(\frac{c_1(p)}{p}t\right) - \frac{c_2(p)}{c_1(p)},$$

By using the inequality

$$L_p(t) = \int_{\Omega} \left(\sum_{i=0}^{i=p} \alpha_i(p) w^i z^{p-i} \right) dx \geq \int_{\Omega} (\alpha_p(p) w^p + \alpha_0(p) z^p) dx,$$

we have

$$(3.13) \quad (L_p(t))^{\frac{1}{p}} \geq \min(\alpha_p(p), \alpha_0(p))^{\frac{1}{p}} \times \sup\left(\left(\int_{\Omega} w^p dx\right)^{\frac{1}{p}}, \left(\int_{\Omega} z^p dx\right)^{\frac{1}{p}}\right),$$

and therefore, for all $t < T^*$,

$$(3.14) \quad \sup(\|w(t, \cdot)\|_p, \|z(t, \cdot)\|_p) \leq \frac{(L_p(t))^{\frac{1}{p}}}{\min(\alpha_p(p), \alpha_0(p))^{\frac{1}{p}}},$$

from (3.12), (3.13) and (3.14), we obtain

$$(3.15) \quad \sup(\|w(t, \cdot)\|_p, \|z(t, \cdot)\|_p) \leq c_p(t), \forall t < T^*,$$

where

$$c_p(t) = \frac{\left[L_p(0) + \frac{c_2(p)}{c_1(p)} \right] \exp\left(\frac{c_1(p)}{p}t\right) - \frac{c_2(p)}{c_1(p)}}{\min(\alpha_p(p), \alpha_0(p))^{\frac{1}{p}}}.$$

The proof of Lemma is complete. \square

Theorem 3.2. *Let $(w(t, \cdot), z(t, \cdot))$ be a solution of the problem (3.1)-(3.4). We assume that the condition (1.6) holds and one of the conditions (1.7) or (1.9) are satisfied. In addition if $\frac{p}{m} > \frac{n}{2}$, then the solution $(w(t, \cdot), z(t, \cdot))$ exists globally in time.*

Proof. From (1.6), we have

$$\begin{aligned} \sup(\|F(w, z)\|_{\frac{p}{m}}^{\frac{p}{m}}, \|G(w, z)\|_{\frac{p}{m}}^{\frac{p}{m}}) &= \sup\left(\left\|(-f + \frac{a - \lambda_1}{c}g)(u, v)\right\|_{\frac{p}{m}}^{\frac{p}{m}}, \left\|(f - \frac{a - \lambda_2}{c}g)(u, v)\right\|_{\frac{p}{m}}^{\frac{p}{m}}\right) \\ &\leq C_m^{\frac{p}{m}} \int_{\Omega} (|u| + v + 1)^p dx \\ &\leq C' \int_{\Omega} (w + z + 1)^p dx \forall t, \quad s \geq 0 \end{aligned}$$

using the following formula

$$\begin{aligned} (3.16) \quad \int_{\Omega} (w + z + 1)^p dx &= \int_{\Omega} \left(\sum_{i=0}^{k=p} C_p^i (w + z)^i \right) dx = \int_{\Omega} ((w + z)^p + 1) dx + \\ &+ \sum_{i=1}^{i=p-1} C_p^i \int_{\Omega} (w + z)^i dx, \end{aligned}$$

An application of Hölder's inequality from (3.16) gives

$$\begin{aligned} (3.17) \quad \int_{\Omega} (w + z + 1)^p dx &\leq \text{mes}(\Omega) + \int_{\Omega} (w + z)^p dx + \\ &+ \sum_{i=1}^{i=p-1} C_p^i (\text{mes}(\Omega))^{\frac{p-i}{p}} \left(\int_{\Omega} (w + z)^p dx \right)^{\frac{i}{p}}, \end{aligned}$$

using (3.15) we get

$$\left(\int_{\Omega} (w + z)^p dx \right)^{\frac{1}{p}} = \|w(t, \cdot) + z(t, \cdot)\|_p \leq \|w(t, \cdot)\|_p + \|z(t, \cdot)\|_p \leq 2c_p(t),$$

and the inequality (3.17) can be written as follows

$$\begin{aligned} \int_{\Omega} (w + z + 1)^p dx &\leq \text{mes}(\Omega) + 2^p (c_p(t))^p + \\ \sum_{i=1}^{i=p-1} C_p^i 2^i (c_p(t))^i (\text{mes}(\Omega))^{\frac{p-i}{p}} &\leq \sum_{i=0}^{i=p} C_p^i 2^i (c_p(t))^i (\text{mes}(\Omega))^{\frac{p-i}{p}}. \end{aligned}$$

□

Therefore

$$\begin{aligned} \sup(\|F(w, z)\|_{\frac{p}{m}}, \|G(w, z)\|_{\frac{p}{m}}) &\leq C'_m \int_{\Omega} (w + z + 1)^p dx \\ &\leq C'_m \sum_{i=0}^{i=p} C_p^i 2^i (c_p(t))^i (\text{mes}(\Omega))^{\frac{p-i}{p}}, \end{aligned}$$

which gives that

$$\sup\left(\left\|(-f + \frac{a - \lambda_1}{c}g)(r, s)\right\|_{\frac{p}{m}}, \left\|(f - \frac{a - \lambda_2}{c}g)(r, s)\right\|_{\frac{p}{m}}\right) \leq C_{p,m}(t), \forall t < T^*, \frac{p}{m} > \frac{n}{2},$$

where

$$C_{p,m}(t) = C'_m \left(\sum_{i=0}^{i=p} C_p^i 2^i (c_p(t))^i (\text{mes}(\Omega))^{\frac{p-i}{p}} \right).$$

If we take $q = \frac{p}{m}$, we obtain a uniform estimate of $\sup(\|(F(w, z)\|_q, \|(F(w, z)\|_q)$, for some $q > \frac{n}{2}$.

This ends the proof of Theorem.

Remark 3.1. It is clear that condition (1.5) implies the positivity of the solution of the system (3.1)-(3.4) on its interval of existence. See [12], for more details.

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REFERENCES

1. N. D. ALIKAKOS: *L_p -Bounds of Solutions of Reaction-Diffusion Equations*. Comm. Partial. Differential. Equations **4** (1979), 827-868.
2. E. L. CUSSLER: *Multicomponent diffusion*. Chemical Engineering Monographs 3. Elsevier Scientific Publishing Company, Amsterdam, 1976.
3. A. HARAUX AND A. YOUKANA: *On a Result of K. Masuda Concerning Reaction-Diffusion Equations*, Tohoku Math. J. **40** (1988), 159-163
4. D. HENRY: *Geometric theory of Semi-Linear Parabolic Equations*, Lecture Notes in Math., 840, Springer Verlas, New York (1983).
5. S. KOUACHI: *Invariant Regions and Global Existence of Ssolutions for Reaction-Diffusion Systems with a Full Matrix of Diffusion Coefficients and Non homogeneous Boundary Conditions*, Georgian Mathematical Journal **11** (2004), 349-359.

6. K. MASUDA: *On the Global Existence and Asymptotic Behavior of Solution of Reaction-Diffusion Equations*, Hokaido Math. J. **12** (1983), pp. 360-370.
7. H. MOHAMED AND Z. DAHMANI: *New results for a coupled system of fractional differential equations*. Facta Universitatis Series: Mathematics and Informatics **28.2** (2013), 133-150.
8. A. MOUMENI AND L. SALAH DERRADJI: *Global Existence of solution for reaction diffusion systems*, IAENG, Int. J. Appl. Math. **40(2)** (2010), 84-90.
9. A. MOUMENI AND L. SALAH DERRADJI: *Global Solution of reaction diffusion system with non diagonal matrix*, Demonstratio Mathematica, Vol. XLV No 1 2012.
10. A. PAZY: *Semigroups of linear operators and applications to partial differential equations*. Applied Math. Sciences, 44:Springer-Verlag, New York, 1983.
11. J. SAVCHIK AND B. CHANGS, AND H. RABITZ: *Application of moments to the general linear multicomponent reaction-diffusion equations*. J. Phys. Chem. **37** (1983), 1990-1997.
12. J. SMOLLER: *Shock Waves and Reaction-Diffusion Equations*, Grundlehren der Mathematischen Wissenschaften, 258 Springer-Verlag, New York (1983).
13. QINGFENG XIAO, XIYAN HU AND LEI ZHANG: *The reflexive extremal rank solutions to the matrix equation $AX = B$* , Facta Universitatis Series: Mathematics and Informatics, **27.1** (2012), 109-115.

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